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The local embedding problem for optical structures

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In this work we examine optical structures and shear-free geometries of rays. Motivated by a paper of Robinson and Trautman and proceeding in analogy with the Chern–Moser construction of the canonical Cartan connection on Cauchy–Riemann manifolds, we obtain special framings adapted to a geometry of rays satisfying a set of equations which are similar to Cartan's equations for Cauchy–Riemann spaces. We show that, given a hypersurface M in the five-dimensional real projective space which is tangent to the vertical distribution of the Hopf fibration, then a shear-free geometry of rays is determined over M. Thereupon, as an application of our construction, we obtain a proof of the following fact: from a local point of view, every real-analytic twisting shear-free geometry of rays can be constructed in this way. We note that the proof of this last assertion can be also obtained by combining a theorem due to Robinson and Trautman and the local embedding theorem for Cauchy–Riemann structures.

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Introduction

An *optical structure* is defined by a four-dimensional manifold M endowed with two plane field distributions $V_0(M)$ and $V_1(M)$ of rank 1 and 3, respectively, such that $V_0(M) \subset V_1(M)$ and by a complex structure on the quotient bundle $V_1(M)/V_0(M)$. These structures were introduced by Robinson and Trautman [8–10] in the study of the so-called radiative solutions of sourceless Maxwell equations (see refs. [5–7]).

A Lorentzian metric on M is said to be *adapted* to the optical structure if $V_0(M)$ is an isotropic sub-bundle of $T(M)$, $V_1(M)$ is its orthogonal complement and the metric tensor induced on $V_1(M)/V_0(M)$ is Hermitian with respect to the complex structure. If we fix an adapted Lorentzian metric and if we find that the integral curves of the characteristic distribution $V_0(M)$ are geodesics, then we will speak of *geometry of rays*. It should be noted that this definition does not depend on the choice of the particular adapted metric. A geometry of rays is said to be *shear free* if the Lie derivative of an adapted metric g in the direction of

$V_0(M)$ is of the form r^2g+h, where r is a smooth function and h is a symmetric two-tensor vanishing on $V_1(M)\times V_1(M)$. Again, this definition is independent of the choice of g.

There are two types of shear-free geometries of rays. The first type is characterized by the complete integrability of the plane field distribution $V_1(M)$. This is equivalent to the integrability in the sense of the theory of G-structures. We refer to refs. [8] and [10] for detailed expositions in this direction. The second type is characterized by the strong non-integrability of $V_1(M)$, that is, the skew-symmetric two-form $V_1(M)\times V_1(M)\to T(M)/V_1(M)$ defined by $(X, Y)=[X, Y]$ mod $V_1(M)$ has everywhere maximal rank. In the literature often the adjective "*twisting*" is used to denote this second case.

In ref. [9], Robinson and Trautman studied the relationships existing between shear-free geometries of rays and Cauchy–Riemann (CR) structures in three dimensions. They showed that, locally, a shear-free geometry of rays is a submersion with one-dimensional fibres over a three-dimensional CR manifold, where the fibres coincide with the rays. Motivated by this result we tried to apply the construction of the canonical Cartan connection of a non-degenerate CR manifold (see refs. [2,3]) to a twisting shear-free geometry of rays. For this reason we introduce the notion of transversal orientation (definition 1.2). Then, given any transversal orientation we construct a suitable framing satisfying a set of equations which have strong similarities with the structure equations of the canonical connection of a three-dimensional CR manifold.

As an application of this construction we examine the local embedding problem for optical structures. Given any hypersurface $f: M\to\mathbb{RP}^5$ which is tangent to the vertical distribution of the Hopf fibration of $\mathbb{RP}^5$ (such hypersurfaces will be called *characteristic*), the immersion f induces on M a shear-free geometry of rays. The local embedding problem that we consider in the present paper can be stated as follows: *Given a shear-free twisting geometry of rays, is it possible to find locally a characteristic immersion in* $\mathbb{RP}^5$ *which induces the original optical structure*? The answer to this problem is positive if the shear-free twisting geometry of rays is real-analytic. We shall prove this fact by using our construction and some basic facts of the theory of exterior differential systems. But we should note that it is possible to derive this result by combining the theorem of Robinson and Trautman and the local embedding theorem for real-analytic CR structures.

The paper is organized as follows: In section 1 we recall the basic definitions and some results concerning optical structures. In section 2 we derive the basic structure equations of a twisting shear-free geometry of rays and in section 3 we construct the normal parallelism. In section 4 we consider a characteristic hypersurface M of $\mathbb{RP}^5$. We exhibit that every characteristic immersion induces on M a shear-free geometry of rays. We prove that every real-analytic shear-free geometry of rays arises in this way (at least locally). During the proof we use some basic theorems of the theory of exterior differential systems (Cartan–Kähler theorem and Cartan's test of involution). Our basic references on this subject are refs. [1] and [4].

1. Optical structures

Definition 1.1. Let M be an oriented four-dimensional manifold. An *optical structure* on M is defined to be a smooth reduction $\pi: Q(M) \to M$ of the linear frame bundle of M to the group

$$H = \left\{ \begin{bmatrix} t & {}^t x & b \\ 0 & A & y \\ 0 & 0 & r \end{bmatrix} \middle| A \in \mathrm{GL}(1, \mathbb{C}) \subset \mathrm{GL}(2, \mathbb{R}), t, b, r \in \mathbb{R}, t, r \neq 0, x, y \in \mathbb{R}^2 \right\}.$$

The principal fibre bundle $Q(M)$ determines two sub-bundles $V_0(M)$ and $V_1(M)$ of $T(M)$, which are defined as follows:

$$V_0(M)_x =: \mathrm{Span}_{\mathbb{R}}(u_0), \qquad V_1(M)_x =: \mathrm{Span}_{\mathbb{R}}(u_0, u_1, u_2),$$

for every x in M and every $u = (x, u_0, u_1, u_2, u_3)$ in $Q(M)_x$. The line $V_0(M)_x$ will be called the *characteristic direction* at x. We define a complex structure J on the vector bundle $V_1(M)/V_0(M)$ by setting $J([u_1]) =: [u_2]$ and $J([u_2]) =: -[u_1]$, where $[\cdot]$ denotes the equivalence class in $V_1(M)/V_0(M)$.

If we indicate by $\theta = {}^t(\theta^0, \theta^1, \theta^2, \theta^3)$ the tautological one-form of $Q(M)$, then the optical structure is said to be a *shear-free geometry of rays* if the one-forms θ^3 and $\omega =: \theta^1 + \mathrm{i}\theta^2$ satisfy the following integrability conditions:

$$\mathrm{d}\theta^3 = \mathrm{i}g\omega \wedge \bar{\omega} + \theta^3 \wedge \alpha, \qquad \mathrm{d}\omega = -\beta \wedge \omega + \gamma \wedge \theta^3, \tag{1.1}$$

where α, β and γ are suitable one-forms and g is a real-valued smooth function.

A shear-free geometry of rays is said to be *twisting* if $g(u) \neq 0$, for every u in $Q(M)$.

It should be quite evident that all the definitions above are equivalent to the ones which can be found in the existing literature (cf. refs. [8–10]). We will leave it to the reader to check the equivalence.

An exterior differential p-form σ ($p = 1, 2$ or 3) is said to be *optical* if

$$i_{V_0(M)}\sigma = 0, \qquad \theta^3 \wedge \pi^*(\sigma) = 0. \tag{1.2}$$

Clearly, σ is an optical two-form iff

$$\pi^*(\sigma) = \mu \wedge \theta^3, \tag{1.3}$$

where μ is a one-form belonging to the span of θ^1, θ^2 and θ^3.

Definition 1.2. Let $Q(M)$ be an optical structure and U be an open set of M. A *transversal orientation* on U is given by a nowhere vanishing optical two-form Θ defined on U and satisfying the following conditions:

(a) $\dim C_1(\Theta)_x = \dim C_2(\Theta)_x = 2$, for every x in M, where $C_i(\Theta)$ denote the sub-bundles of $T^*(U)$ given by

$$C_1(\Theta) = \{i_V \mathrm{Re}(\Theta) / V \in T(U)\},$$
$$C_2(\Theta) = \{i_V \mathrm{Im}(\Theta) / V \in T(U)\}.$$

(b) $\dim C_1(\Theta) \oplus C_2(\Theta) = 3$ and the annihilator of $C_1(\Theta) \oplus C_2(\Theta)$ is the characteristic distribution $V_0(M)|_U$.

(c) $C_1(\Theta) \cap C_2(\Theta)$ is the line sub-bundle of all one-forms annihilating the three-dimensional plane field distribution $V_1(M)|_U$. Moreover,

$$\mathrm{d}(C_1(\Theta) \cap C_2(\Theta)) \neq 0 \mod T^*(U) \wedge (C_1(\Theta) \cap C_2(\Theta)).$$

This means that for every nowhere vanishing local cross-section γ of $C_1(\Theta) \cap C_2(\Theta)$, the equivalence class determined by $\mathrm{d}\gamma$ in $\wedge^2 T^*(U)/T^*(U) \wedge (C_1(\Theta) \cap C_2(\Theta))$ is everywhere $\neq 0$ whenever this makes sense, that is, in the domain of definition of the one-form γ.

(d) $$i_X \,\mathrm{d}\Theta = \mathrm{i}k(X)\Theta,$$

for every cross-section X of $V_0(M)|_U$, where k is a nowhere vanishing cross-section of the vector bundle $\mathrm{Hom}(V_0(M)|_U, \mathbb{R})$.

Proposition 1.3. *Let $Q(M) \to M$ be an optical structure and x be any point of M. Then there exists an open neighbourhood U of x and a transversal orientation Θ defined on U. Moreover, if $Q(M) \to M$ is real-analytic then we may choose Θ to be real-analytic.*

Proof. Let $x \in M$ be given. We fix a simply connected open neighbourhood U of x, a local coordinate system $(x^0, \ldots, x^3)$ defined on U and a local cross-section $u: U \to Q(M)$ such that $\theta^0(\partial/\partial x^0) = F$, where $F(y) \neq 0$, and $\theta^a(\partial/\partial x^0)|_y = 0$, $a = 1, 2, 3$, for every $y \in U$. Here we will drop pull-back notations to indicate forms on $Q(M)|_U$ and their pull-backs on U via the cross-section u. Since our optical geometry is shear-free, we then have $\mathrm{d}(\omega \wedge \theta^3) = (A\theta^0 + B\bar{\omega}) \wedge (\omega \wedge \theta^3)$, where A and B are smooth functions defined on U. Let us now indicate by f the function

$$f(y) =: \int_{x^0(x)}^{x^0(y)} [\mathrm{i} - (AF)(x^0, x^1(y), x^2(y), x^3(y))] \,\mathrm{d}x^0,$$

then $\partial f/\partial x^0 = \mathrm{i} - AF$. This implies that the two-form $\Theta =: \mathrm{e}^f \omega \wedge \theta^3$ is a transversal orientation. If $Q(M)$ is real-analytic, then we may choose the cross-section u to be real-analytic as well as the coordinate system $(x^0, ..., x^3)$. Then A, F and f are real-analytic and hence Θ is real-analytic. □

From now on we will assume that $Q(M)$ is a real-analytic shear-free geometry of rays and that Θ is a real-analytic transversal orientation on M. We define a real-analytic reduction $P_0(M, \Theta)$ of $Q(M)$ by setting

$$P_0(M, \Theta) = \{u \in Q(M) \mid \pi^*(\Theta)|_u = (\theta^1 + i\theta^2)|_u \wedge \theta^3|_u\}, \tag{1.4}$$

where $\theta = {}^t(\theta^0, \theta^1, \theta^2, \theta^3)$ is the tautological one-form of $Q(M)$. The structure group H_0 of $P_0(M, \Theta)$ is given by all 4 by 4 matrices of the form

$$A = \begin{bmatrix} t & {}^t x & b \\ 0 & \mathrm{id}_{2\times 2}/r & y \\ 0 & 0 & r \end{bmatrix},$$

where t, b, r are real numbers, $r, t \neq 0$ and $x, y \in \mathbb{R}^2$. For brevity, the element A will be denoted by the symbol $A(y, x, r, t, b)$. On several occasions we identify an element $y = {}^t(y^1, y^2)$ of $\mathbb{R}^2$ with the corresponding complex number $y^1 + iy^2$.

If we perform a right translation by an element A of the structure group, then the tautological one-form behaves as follows:

$$\begin{aligned} &R_A^*(\theta^3) = \frac{1}{r}\theta^3, \qquad R_A^*(\omega) = r\omega - y\theta^3, \\ &R_A^*(\theta^0) = \frac{1}{t}\left(\theta^0 - rx_1\theta^1 - rx_2\theta^2 - \frac{1}{r}(b - rx_1y^1 - rx_2y^2)\theta^3\right), \end{aligned} \tag{1.5}$$

for every $A(y, x, r, t, b)$. Here ω stands for $\theta^1 + i\theta^2$. From (c) and (d) of definition 1.2, it follows that

$$d\theta^3 = ig\omega \wedge \bar{\omega} + \theta^3 \wedge \alpha, \tag{1.6}$$

where α is a suitable one-form and g is a nowhere vanishing real-valued function. Moreover, (1.5) implies

$$g(uA) = \frac{1}{r^3} g(u), \quad \forall u \in P_0(M, \Theta), \forall A \in H_0. \tag{1.7}$$

We define a real-analytic reduced sub-bundle of $P_0(M, \Theta)$ by setting

$$P_1(M, \Theta) = \{u \in P_0(M, \Theta) \mid g(u) = 1\}. \tag{1.8}$$

The structure group H_1 is now given by all 4 by 4 matrices $A(y, x, r, t, b) \in H_0$ such that $r = 1$. For brevity we will indicate by $A(y, x, t, b)$ a generic element of H_1.

2. The structure equations

Let $\tau=(\lambda, \phi^1, \phi, \psi)$ be a real-analytic $\mathbb{C}^2\oplus\mathbb{R}^2$-valued exterior differential one-form, defined on an open subset of $P_1(M, \Theta)$. We say that τ is a *one-form of the first kind* if the following conditions are satisfied:

$$\lambda \text{ is semi-basic}, \quad \phi=\lambda+\bar{\lambda} \text{ does not involve } \theta^0 \text{ and } \theta^3, \tag{2.1a}$$

$$d\theta^3=i\omega\wedge\bar{\omega}+\theta^3\wedge\phi, \tag{2.1b}$$

$$d\omega=-\lambda\wedge\omega+\theta^3\wedge\phi^1, \tag{2.1c}$$

$$d\phi=i\phi^1\wedge\bar{\omega}-i\overline{\phi^1}\wedge\omega+\theta^3\wedge\psi, \tag{2.1d}$$

$$\mathrm{Im}(\lambda)\wedge\omega\wedge\bar{\omega}\wedge\theta^3|_u\neq 0, \quad \forall u\in P_1(M, \Theta). \tag{2.1e}$$

Proposition 2.1. *For every $u\in P_1(M, \Theta)$ there is an open neighbourhood U of u and a one-form of the first kind defined on U.*

Proof. Let α, β and γ be real-analytic one-forms, defined near u, such that

$$d\theta^3=i\omega\wedge\bar{\omega}+\theta^3\wedge\alpha, \tag{i}$$

$$d\omega=-\beta\wedge\omega+\theta^3\wedge\gamma. \tag{ii}$$

Since the one-form θ^3 is projectable as well as its differential it follows that α is semi-basic. Similarly for $\omega\wedge\theta^3$, and hence $\beta+\alpha$ is again semi-basic. This implies that β is semi-basic. Therefore there exist real-analytic functions A and C such that

$$\beta+\bar{\beta}-\alpha=A\omega+\bar{A}\bar{\omega}+C\theta^3. \tag{iii}$$

We define α', β' and γ' by setting

$$\alpha=\alpha', \qquad \beta'=\beta-A\omega-\tfrac{1}{2}C\theta^3, \qquad \gamma=\gamma'+\tfrac{1}{2}C\omega.$$

These one-forms satisfy

$$d\theta^3=i\omega\wedge\bar{\omega}+\theta^3\wedge\alpha, \tag{iv}$$

$$d\omega=-\beta'\wedge\omega+\theta^3\wedge\gamma', \tag{v}$$

$$\beta'+\bar{\beta}'=\alpha'. \tag{vi}$$

Since α' is semi-basic, then we may write $\alpha'=A_i\theta^i$, where $A_0, \ldots, A_1$ are real-analytic functions. We define λ, ϕ^1, and ϕ by setting

$$\phi=\alpha'-A_3\theta^3, \qquad \lambda=\beta'-\tfrac{1}{2}A_3\theta^3, \qquad \phi^1=\gamma'-\tfrac{1}{2}A_3\omega.$$

The forms ϕ, λ, ϕ^1 satisfy

$$d\theta^3 = i\omega\wedge\bar\omega + \theta^3\wedge\phi\,, \tag{vii}$$

$$d\omega = -\lambda\wedge\omega + \theta^3\wedge\phi^1\,, \tag{viii}$$

$$\lambda+\bar\lambda=\phi\,,\quad \lambda \text{ is semi-basic and } \phi \text{ does not involve } \theta^3\,. \tag{ix}$$

From (vii) and (viii), and using (c) and (d) in definition 1.2, it is easily seen that the form ϕ does not involve θ^0 and the form λ satisfies (e) of (2.1). Differentiating (vii) and using (ix) we obtain

$$(d\phi - i\phi^1\wedge\bar\omega + i\overline{\phi^1}\wedge\omega)\wedge\theta^3 = 0\,.$$

Therefore there exists a one-form ψ such that

$$d\phi = i\phi^1\wedge\bar\omega - i\overline{\phi^1}\wedge\omega + \theta^3\wedge\psi\,.$$

From all this we conclude that $(\lambda, \phi^1, \phi, \psi)$ is a one-form of the first kind. □

Lemma 2.2. *Let τ and τ' be one-forms of the first kind defined on the open sets U and U', respectively. Then, on the overlap $U\cap U'$ there exist real-analytic functions D, E and G, where $D=-\bar D$, such that*

$$\phi=\phi'\,, \tag{i}$$

$$\lambda=\lambda'+D\theta^3\,, \tag{ii}$$

$$\phi^1=\phi^{1\prime}+D\omega+E\theta^3\,, \tag{iii}$$

$$\psi=\psi'+i(\bar E\omega-E\bar\omega)+G\theta^3\,. \tag{iv}$$

Proof. The form $\phi-\phi'$ is semi-basic and does not involve θ^3. From (2.1b) we get $(\phi-\phi')\wedge\theta^3=0$, thus ϕ and ϕ' coincide. From (2.1c) it follows that $-(\lambda-\lambda')\wedge\omega+\theta^3\wedge(\phi^1-\phi^{1\prime})=0$. Using Cartan's lemma we obtain

$$\begin{aligned}\lambda-\lambda' &= D\theta^3+\hat D\omega\,,\\ \phi^1-\phi^{1\prime} &= D\omega+E\theta^3\,.\end{aligned} \tag{2.2}$$

From (2.1a) we have $(\lambda-\lambda')+(\overline{\lambda-\lambda'})=0$; this implies $\hat D=0$ and $D=-\bar D$. Equation (2.1d) gives $\theta^3\wedge(\psi-\psi'+iE\bar\omega-i\bar E\omega)=0$; then $\psi-\psi'+iE\bar\omega-i\bar E\omega=G\theta^3$, for some real-analytic function G. □

Given a one-form of the first kind τ we define Λ to be the complex-valued two-form

$$\Lambda = d\lambda - i\bar\omega\wedge\phi^1 + 2i\overline{\phi^1}\wedge\omega\,. \tag{2.3}$$

Lemma 2.3. *The two-form Λ satisfies*

$$\Lambda = S\omega \wedge \bar{\omega} \mod \theta^3 , \tag{2.4}$$

where S is a real-analytic real-valued function.

Proof. Differentiating $\lambda + \bar{\lambda} = \phi$ we get

$$(\Lambda + \mathrm{i}\bar{\omega} \wedge \phi^1 - 2\mathrm{i}\overline{\phi^1} \wedge \omega) + (\bar{\Lambda} - \mathrm{i}\omega \wedge \overline{\phi^1} + 2\mathrm{i}\phi^1 \wedge \bar{\omega}) = \mathrm{i}\phi^1 \wedge \bar{\omega} - \mathrm{i}\overline{\phi^1} \wedge \omega + \theta^3 \wedge \phi ;$$

therefore $\Lambda + \bar{\Lambda} = \theta^3 \wedge \psi$. Differentiating $\mathrm{d}\omega = \mathrm{i}\lambda \wedge \omega + \theta^3 \wedge \phi^1$ we get $\Lambda \wedge \omega = 0$ mod θ^3. The last two equations imply the required result. □

An *exterior differential one-form of the second kind* is defined to be a one-form of the first kind such that $S = 0$.

Proposition 2.4. *For every $u \in P_1(M, \Theta)$ there is an open neighbourhood U on which an exterior differential form of the second kind is defined.*

Proof. Let $\tau = (\lambda, \phi^1, \phi, \psi)$ be a one-form of the first kind, D and E be real-analytic functions such that $D = -\bar{D}$. Then the form $\tau' = (\lambda', \phi^{1\prime}, \phi', \psi')$ defined by

$$\lambda' = \lambda - D\theta^3 , \qquad \phi^{1\prime} = \phi^1 - D\omega - E\theta^3 , \qquad \psi' = \psi - \mathrm{i}(\bar{E}\omega - E\bar{\omega}) ,$$

is a one-form of the first kind. Since Λ and Λ' satisfy $\Lambda = \Lambda' - 4\mathrm{i}\bar{D}\omega \wedge \bar{\omega}$ mod θ^3, then $S = S' - 4\mathrm{i}\bar{D}$. Therefore, if we set $D = -\frac{1}{4}\mathrm{i}S$ and $E = 0$, then $S' = 0$ and this gives the required result. □

Lemma 2.5. *Let τ and τ' be one-forms of the second kind. Then on the intersection of their domains of definition there are real-analytic functions E and G such that*

$$\phi = \phi' , \qquad \lambda = \lambda' , \tag{2.5a}$$

$$\phi^1 = \phi^{1\prime} + E\theta^3 , \tag{2.5b}$$

$$\psi = \psi' + \mathrm{i}(\bar{E}\omega - E\bar{\omega}) + G\theta^3 . \tag{2.5c}$$

The proof of this lemma follows immediately from the proof of lemma 2.2 and is left to the reader.

Lemma 2.6. *Let τ be a one-form of the second kind, then*

$$\Lambda = V\omega \wedge \theta^3 - \bar{V}\bar{\omega} \wedge \theta^3 - \tfrac{1}{2}\psi \wedge \theta^3 , \tag{2.6}$$

where V is a real-analytic complex-valued function.

Proof. Lemma 2.3 implies the existence of a one-form ρ such that $\Lambda=\rho\wedge\theta^3$ and $2\mathrm{Re}(\rho)+\psi=0 \mod \theta^3$. Differentiation of $\mathrm{d}\omega=-\lambda\wedge\omega+\theta^3\wedge\phi^1$ gives

$$\mathrm{d}\phi^1=-\lambda\wedge\phi^1+\rho\wedge\omega+\phi\wedge\phi^1+\mu\wedge\theta^3\,, \tag{i}$$

where μ is a suitable one-form. Differentiating $\rho\wedge\theta^3=\mathrm{d}\lambda-\mathrm{i}\bar{\omega}\wedge\phi^1+2\mathrm{i}\overline{\phi^1}$ and using (i) we obtain

$$2\mathrm{i}(\bar{\rho}-\rho)\wedge\omega\wedge\bar{\omega}=0 \quad \mod \theta^3\,, \tag{ii}$$

from which we get $(2\rho+\psi)\wedge\omega\wedge\bar{\omega}=0 \mod \theta^3$ and this implies

$$\rho=-\tfrac{1}{2}\psi+V\omega+W\bar{\omega}+A\theta^3\,,$$

where V, W and A are complex-valued real-analytic functions. Since $2\mathrm{Re}(\rho)+\psi=0$, then $W=-\bar{V}$ and this conclusion gives the proof of the lemma.

□

A one-form of the second kind satisfying $V=0$ is said to be a *one-form of the third kind.*

Proposition 2.7. *For every $u\in P_1(M, \Theta)$ there is an open neighbourhood on which a one-form of the third kind is defined.*

Proof. Let τ and τ' be exterior differential forms of the second kind defined on U and U', respectively. Then, using the same notations as in lemma 2.5 we get from (2.3) $\Lambda=\Lambda'-\mathrm{i}E\bar{\omega}\wedge\theta^3-2\mathrm{i}\bar{E}\omega\wedge\theta^3$, which implies $V=V'-\frac{3}{2}\mathrm{i}\bar{E}$. Therefore, given τ', if we define τ by

$$\phi=\phi'\,,\qquad \lambda=\lambda'\,,\qquad \phi^1=\phi^{1\prime}+\tfrac{4}{3}\mathrm{i}\bar{V}'\theta^3\,,$$
$$\psi=\psi'+\tfrac{2}{3}(V'\omega+\bar{V}'\bar{\omega})\,,$$

then we will get a one-form of the third kind. □

Lemma 2.8. *Let τ and τ' be exterior differential forms of the third kind defined on open sets U and U', respectively. Then, on the overlap $U\cap U'$ we have*

$$\phi=\phi'\,,\qquad \lambda=\lambda'\,,\qquad \phi^1=\phi^{1\prime}\,,\qquad \psi=\psi'+G\theta^3\,, \tag{2.7}$$

where G is a real-analytic real-valued function.

The proof of this lemma follows immediately from the proof of proposition 2.7. The details are left to the reader.

Given a one-form of the third kind we define Φ^1 and Ψ to be the exterior-differential two-forms given by

$$\Phi^1 = \mathrm{d}\phi^1 + \phi^1 \wedge \phi + \lambda \wedge \phi^1 + \tfrac{1}{2}\psi \wedge \omega\,, \qquad (2.8)$$
$$\Psi = \mathrm{d}\psi - \phi \wedge \psi - 2\mathrm{i}\phi^1 \wedge \overline{\phi^1}\,.$$

Lemma 2.9. *The two-form Φ^1 is semi-basic and it can be expressed by*

$$\Phi^1 = (P\omega + Q\bar{\omega}) \wedge \theta^3\,, \qquad (2.9)$$

where P and Q are real-analytic complex-valued functions.

Proof. Differentiating $\mathrm{d}\omega = -\lambda \wedge \omega + \theta^3 \wedge \phi^1$ we get

$$\theta^3 \wedge (\mathrm{d}\phi^1 + \phi^1 \wedge \phi + \lambda \wedge \phi^1 - \tfrac{1}{2}\psi \wedge \omega) = 0\,.$$

This implies that $\Phi^1 = \nu \wedge \theta^3$, where ν is a complex-valued one-form. Differentiation of $\mathrm{d}\phi = \mathrm{i}\phi^1 \wedge \bar{\omega} - \mathrm{i}\overline{\phi^1} \wedge \omega + \theta^3 \wedge \psi$ gives

$$\theta^3 \wedge (\Psi + \mathrm{i}\nu \wedge \bar{\omega} - \mathrm{i}\bar{\nu} \wedge \omega) = 0\,. \qquad (2.10)$$

Therefore we obtain

$$\mathrm{d}\psi = \phi \wedge \psi + 2\mathrm{i}\phi^1 \wedge \overline{\phi^1} + \mathrm{i}\bar{\nu} \wedge \omega - \mathrm{i}\nu \wedge \bar{\omega} + \rho \wedge \theta^3\,, \qquad \text{(i)}$$

where ρ is a real-valued one-form. Differentiating $\mathrm{d}\phi^1 + \phi^1 \wedge \phi + \lambda \wedge \phi^1 - \nu \wedge \theta^3 + \tfrac{1}{2}\psi \wedge \omega = 0$ and using (i) we deduce

$$-\tfrac{3}{2}\nu \wedge \bar{\omega} \wedge \omega + \theta^3 \wedge (-\mathrm{d}\nu - 2\nu \wedge \phi + \nu \wedge \lambda - \tfrac{1}{2}\rho \wedge \omega) = 0\,. \qquad (2.11)$$

This last equation implies that ν is semi-basic and does not involve θ^0, and hence the lemma is proved. □

An *exterior differential form of the fourth kind* is defined to be a one-form of the third kind such that the real part of the function P vanishes identically.

Proposition 2.10. *There exists a unique globally defined one-form of the fourth kind.*

Proof. If τ and τ' are one-forms of the third kind defined on U and U', respectively, then on $U \cap U'$ there is a real-analytic function G such that

$$\phi = \phi'\,, \qquad \lambda = \lambda'\,, \qquad \phi^1 = \phi^{1\prime}\,, \qquad \psi = \psi' + G\theta^3\,.$$

From this we have $\Phi^1 = \Phi^{1\prime} - \tfrac{1}{2}G\omega \wedge \theta^3$. In particular, given τ' we define τ by

$$\phi = \phi'\,, \qquad \lambda = \lambda'\,, \qquad \phi^1 = \phi^{1\prime}\,, \qquad \psi = \psi' + 2\mathrm{Re}(P')\theta^3\,.$$

Clearly, τ is a one-form of the fourth kind and this shows the local existence. Moreover, if τ and τ'' are one-forms of the fourth kind it is also clear that τ and τ'' must coincide on the overlap of their domains of definition. This implies the global existence and the uniqueness. □

Proposition 2.11. *Let τ be the one-form of the fourth kind, then*

$$\Phi^1=Q\bar{\omega}\wedge\theta^3,\qquad \Psi=R\omega\wedge\theta^3+\bar{R}\bar{\omega}\wedge\theta^3, \tag{2.12}$$

where Q and R are real-analytic complex-valued functions.

Proof. We know that $\Phi^1=(\mathrm{i}T\omega\wedge\theta^3+Q\bar{\omega})\wedge\theta^3$, where T is real-valued and Q is complex-valued. Substitution in (2.10) gives $\Psi=0 \bmod \theta^3$. Hence we obtain

$$\mathrm{d}\psi=\phi\wedge\psi+2\mathrm{i}\phi^1\wedge\overline{\phi^1}+\rho\wedge\theta^3. \tag{i}$$

Differentiating $\mathrm{d}\lambda=-\frac{1}{2}\psi\wedge\theta^3+\mathrm{i}\bar{\omega}\wedge\phi^1-2\mathrm{i}\overline{\phi^1}\wedge\omega$ and using (i) we get $3T\bar{\omega}\wedge\omega\wedge\theta^3=0$, which implies $\Phi^1=Q\bar{\omega}\wedge\theta^3$. Differentiation of (i) gives $\rho=0 \bmod(\theta^3, \omega, \bar{\omega})$, from which we obtain the second equation of (2.12). □

Now, summarizing the results of the preceding discussion we may formulate our first theorem.

Theorem 2.12. *Let $Q(M)$ be a real-analytic shear-free geometry of rays equipped with a real-analytic transversal orientation Θ. Then the principal fibre bundle $P_1(M,\Theta)$ has a unique real-analytic $\mathbb{C}^2\oplus\mathbb{R}^2$-valued one-form $\tau=(\lambda,\phi^1,\phi,\psi)$ satisfying the following equations:*

$$\lambda=\mathrm{i}p\theta^0+q\omega+r\bar{\omega}+\mathrm{i}s\theta^3, \tag{2.13a}$$

$$\phi=(q+\bar{r})\omega+(r+\bar{q})\bar{\omega}, \tag{2.13b}$$

$$\mathrm{d}\theta^3=\mathrm{i}\omega\wedge\bar{\omega}+\theta^3\wedge\phi, \tag{2.13c}$$

$$\mathrm{d}\omega=-\lambda\wedge\omega+\theta^3\wedge\phi^1, \tag{2.13d}$$

$$\mathrm{d}\phi=\mathrm{i}\phi^1\wedge\bar{\omega}-\mathrm{i}\overline{\phi^1}\wedge\omega+\theta^3\wedge\psi, \tag{2.13e}$$

$$\mathrm{d}\lambda=-\tfrac{1}{2}\psi\wedge\theta^3+\mathrm{i}\bar{\omega}\wedge\phi^1-2\mathrm{i}\overline{\phi^1}\wedge\omega, \tag{2.13f}$$

$$\mathrm{d}\phi^1=Q\bar{\omega}\wedge\theta^3-\phi^1\wedge\phi-\lambda\wedge\phi^1-\tfrac{1}{2}\psi\wedge\omega, \tag{2.13g}$$

$$\mathrm{d}\psi=R\omega\wedge\theta^3+\bar{R}\bar{\omega}\wedge\theta^3+\phi\wedge\psi+2\mathrm{i}\phi^1\wedge\overline{\phi^1}, \tag{2.13h}$$

where p, q, r, s, Q and R are real-analytic functions, all of them are complex-valued except for p and s which are real-valued. Moreover, the function p is nowhere vanishing.

3. The normal coframing

Now we investigate the transformation rules of the function $q+\bar{r}$. Using (1.5) and (2.13c) we have

$$\mathrm{i}\omega\wedge\bar{\omega}+\theta^3\wedge\phi=\mathrm{i}R_A^*(\omega)\wedge R_A^*(\bar{\omega})+\theta^3\wedge R_A^*(\phi)$$
$$=\mathrm{i}(\omega-\dot{y}\theta^3)\wedge(\bar{\omega}-\bar{y}\theta^3)+\theta^3\wedge R_A^*(\phi)\,,$$

for every $A=A(y,x,t,b)\in H_1$. This implies

$$R_A^*(\phi)=\phi-\mathrm{i}\bar{y}\omega-\mathrm{i}y\bar{\omega}\quad\mathrm{mod}\,\theta^3\,. \tag{3.1}$$

On the other hand

$$R_A^*(\phi)_u=(q+\bar{r})(uA)(\omega-y\theta^3)+(\bar{q}+r)(uA)(\bar{\omega}-\bar{y}\theta^3)\,.$$

Thus, we obtain

$$(q+\bar{r})(uA)=(q+\bar{r})(u)-\mathrm{i}\bar{y}\,, \tag{3.2}$$

for every $u\in P_1(M,\Theta)$ and every $A\in H_1$. Using (3.2) we reduce the principal fibre bundle $P_1(M,\Theta)$:

$$P_2(M,\Theta)=\{u\in P_1(M,\Theta)\mid(q+\bar{r})_u=0\}\,. \tag{3.3}$$

From (3.2) it is clear that $P_2(M,\Theta)$ is a real-analytic reduced sub-bundle with structure group

$$H_2=\left\{A(x,t,b)=\begin{bmatrix}t & {}^{\mathrm{t}}x & b\\ 0 & \mathrm{id}_{2\times2} & 0\\ 0 & 0 & 1\end{bmatrix}\,\middle|\, t,b\in\mathbb{R},\,t\neq0,\,x\in\mathbb{R}^2\right\}.$$

On $P_2(M,\Theta)$ we have

$$\phi=0\,,\qquad\lambda=\mathrm{i}p\theta^0+q\omega-\bar{q}\bar{\omega}+\mathrm{i}s\theta^3\,. \tag{3.4}$$

Using (2.13d) and (1.5) we get

$$[\lambda-R_A^*(\lambda)]\wedge\omega+\theta^3\wedge[R_A^*(\phi^1)-\phi^1]=0\,, \tag{3.5}$$

for every $A\in H_2$. From (3.5) we get

$$p(u)=\frac{1}{t}p(uA)\,,$$

$$2q_2(u)=2q_2(uA)-p(uA)\frac{x_1}{t}\,, \tag{3.6}$$

$$2q_1(u)=2q_1(uA)-p(uA)\frac{x_2}{t}\,,$$

for every $u\in P_2(M,\Theta)$ and $A(x,t,b)\in H_2$, where $x=x_1+\mathrm{i}x_2$ and $q=q_1+\mathrm{i}q_2$. Using (3.6) we obtain that

$$P_3(M,\Theta)=\{u\in P_2(M,\Theta)\mid p(u)=1\,,\,q(u)=0\} \tag{3.7}$$

is a real-analytic reduced sub-bundle of $P_2(M,\Theta)$ whose structure group is given by

$$H_3=\left\{A(b)=\begin{bmatrix}1 & 0 & b\\ 0 & \mathrm{id}_{2\times 2} & 0\\ 0 & 0 & 1\end{bmatrix}\,\middle|\, b\in\mathbb{R}\right\}.$$

On $P_3(M, \Theta)$ we have

$$\phi=0\,,\qquad \lambda=\mathrm{i}\theta^0+\mathrm{i}s\theta^3\,. \tag{3.8}$$

From (3.8) and (2.13e) we deduce that ϕ^1 and ψ are semi-basic one-forms along $P_3(M, \Theta)$. Therefore we may write

$$\phi^1=\mathrm{i}a\omega+B\bar{\omega}+\mathrm{i}\bar{D}\theta^3\,, \tag{3.9a}$$

$$\psi=D\omega+\bar{D}\bar{\omega}+E\theta^3\,, \tag{3.9b}$$

where a and E are real-valued smooth functions and B, D are complex-valued. Using again (3.5) we obtain

$$(s-a)(u)-(s-a)(uA)+b=0\,, \tag{3.10}$$

for every $u\in P_3(M, Q)$ and every $A(b)\in H_3$. As a consequence of (3.10) we see that

$$P_4(M, \Theta)=\{u\in P_3(M, \Theta)\mid (s-a)_u=0\} \tag{3.11}$$

defines a real-analytic absolute parallelism, which will be called the *normal parallelization* of (M, Θ).

Theorem 3.1. *Let M be a four-dimensional manifold and Θ be a regular optical two-form. Then M admits a unique global coframing $(\omega^0, \omega^1, \omega^2, \omega^3)$ satisfying the following structure equations:*

$$\Theta=(\omega^1+\mathrm{i}\omega^2)\wedge\omega^3\,, \tag{3.12a}$$

$$\mathrm{d}\omega^3=\mathrm{i}\omega\wedge\bar{\omega}\,, \tag{3.12b}$$

$$\mathrm{d}\omega^1=-3\omega^0\wedge\omega^2-B_1\omega^1\wedge\omega^3+(s-B_2)\omega^2\wedge\omega^3\,, \tag{3.12c}$$

$$\mathrm{d}\omega^2=3\omega^0\wedge\omega^1-(s+B_2)\omega^1\wedge\omega^3+B_1\omega^2\wedge\omega^3\,, \tag{3.12d}$$

$$\mathrm{d}\omega^0=2s\omega^1\wedge\omega^2-(D_2\omega^1+D_1\omega^2)\wedge\omega^3\,, \tag{3.12e}$$

where s, B_1, B_2, D_1 and D_2 are suitable smooth functions [*uniquely determined by* (3.12)]. *This coframing it said to be the normal coframing of the regular optical two-form Θ.*

Proof. The existence of such a global coframing is a direct consequence of section 2. Let $(\theta^0, \ldots, \theta^3)$ be the coframing determined by the normal absolute parallelism and $\tau=(\lambda, \phi^1, \phi=0, \psi)$ be the one-form of the fourth kind. Then

$$\lambda = \mathrm{i}(\theta^0 + s\theta^3)\,,$$
$$\phi^1 = \mathrm{i}s\omega + B\bar{\omega} + \mathrm{i}\bar{D}\theta^3\,,$$
$$\psi = D\omega + \bar{D}\bar{\omega} + E\theta^3\,,$$

where s, $B=B_1+\mathrm{i}B_2$, $D=D_1+\mathrm{i}D_2$ and E are smooth functions. Define $\omega^0, ..., \omega^3$ by

$$\omega^0 = -\tfrac{1}{3}(\theta^0 + s\theta^3)\,, \qquad \omega^a = \theta^a\,, \quad a=1,2,3\,.$$

Then (3.12e) follows from the definition, (3.12d) is a consequence of (2.13c). Similarly, (3.12a) can be obtained from (2.13f) and (3.12b,c) can be deduced from (2.13d). This concludes the proof of the existence. Take any coframing $(\omega^0, ..., \omega^3)$ satisfying (3.12) and define

$$\theta^0 = -(3\omega^0 + s\omega^3)\,, \qquad \theta^a = \omega^a\,, \quad a=1,2,3\,,$$
$$\lambda = -3\mathrm{i}\omega^0\,, \qquad \phi^1 = \mathrm{i}s\omega + B\bar{\omega} + \mathrm{i}\bar{D}\theta^3\,, \qquad \psi = D\omega + \bar{D}\bar{\omega} + E\theta^3\,,$$

where E is given by

$$E = 2(s^2 - B\bar{B} + \tfrac{1}{2}u_1(D_2) + \tfrac{1}{2}u_2(D_1))\,,$$

where $(u_0, ..., u_3)$ is the framing dual to $(\theta^0, ..., \theta^3)$. We indicate by $K(M)$ the corresponding absolute parallelism. From the definition and from (3.12) we deduce that $K(M)$ is a reduction of $P_1(M, \Theta)$ and $\tau =: (\lambda, \phi^1, \phi=0, \psi)$ is the restriction along $K(M)$ of the one-form of the fourth kind. Since $\lambda = \mathrm{i}(\theta^0 + s\theta^3)$ and $\phi^1 = \mathrm{i}s\omega + B\bar{\omega} + \mathrm{i}\bar{D}\theta^3$, then $K(M)$ coincides with the normal parallelization. From all this it follows that $(\omega_0, ..., \omega_3)$ is uniquely determined. □

4. The local embedding theorem

Consider the five-dimensional real projective space $\mathbb{RP}^5$ as a homogeneous space of the Lie group $\mathrm{SL}(3, \mathbb{C})$ with isotropy subgroup

$$K = \left\{ \begin{bmatrix} r & a_1 \;\; a_2 \\ 0 & A \end{bmatrix} \middle|\, A \in \mathrm{GL}(2, \mathbb{C}),\, r \in \mathbb{R},\, r\det(A) = 1,\, a_1, a_2 \in \mathbb{C} \right\}.$$

The group $\mathrm{SL}(3, \mathbb{C})$ is the total space of a principal K-fibre fundle over $\mathbb{RP}^5$ and we indicate by p the bundle map. The Maurer–Cartan one-form of $\mathrm{SL}(3, \mathbb{C})$ is denoted by $\zeta = (\zeta^a_b)_{a,b=0,1,2}$, and α, β are the real and imaginary part of ζ, respectively. The set of one-forms $\{\beta^0_0, \alpha^1_0, \alpha^2_0, \beta^1_0, \beta^2_0\}$ are linearly independent and generate the space of semi-basic one-forms. The transformation rules of ζ under the adjoint representation give

$$R_X^* \begin{vmatrix} \zeta_0^1 \\ \zeta_0^2 \end{vmatrix} = rA^{-1} \begin{vmatrix} \zeta_0^1 \\ \zeta_0^2 \end{vmatrix},$$

for every $X \in K$. Then $\{\alpha_0^1, \alpha_0^2, \beta_0^1, \beta_0^2\}$ generates a one-dimensional plane field distribution $\mathscr{V}$ on $\mathbb{RP}^5$; this is the vertical distribution of the Hopf fibration of $\mathbb{RP}^5$ over $\mathbb{CP}^2$.

Let $f: M \to \mathbb{RP}^5$ be a hypersurface of $\mathbb{RP}^5$ which is everywhere tangent to the vertical distribution $\mathscr{V}$ (i.e., $\mathscr{V}_{f(x)} \subset \mathrm{d}f[T_x(M)]$, for every $x \in M$). For brevity we say that (M, f) is a *characteristic hypersurface* of $\mathbb{RP}^5$. Consider now the pull-back bundle $\mathscr{F}_0(f) = f^*(\mathrm{SL}(3, \mathbb{C}))$. From now on we will drop pull-back notations for exterior-differential forms. Since we assume f to be tangent to the vertical distribution, then

$$\mathscr{F}_1(f) =: \{u \in \mathscr{F}_0(f) \mid \beta_0^2|_u = 0\} \tag{4.1}$$

is a reduced sub-bundle of $\mathscr{F}_0(f)$ with structure group

$$K_1 = \left\{ \begin{bmatrix} r & a_1 & a_2 \\ 0 & 1/rs & a_2^1 \\ 0 & 0 & s \end{bmatrix} \middle| \; r, s \in \mathbb{R}, r, s \neq 0, a_1, a_2, a_2^1 \in \mathbb{C} \right\}.$$

From the structure equations $\mathrm{d}\zeta = -\zeta \wedge \zeta$ we obtain

$$\mathrm{d}\zeta_0^2 = -\zeta_0^2 \wedge \zeta_0^0 - \zeta_1^2 \wedge \zeta_0^1 - \zeta_2^2 \wedge \zeta_0^2. \tag{4.2}$$

Since ζ_0^2 is real-valued, we have

$$2\mathrm{i}(\beta_0^0 - \beta_2^2) \wedge \alpha_0^2 - \zeta_1^2 \wedge \zeta_0^1 + \overline{\zeta_1^2} \wedge \overline{\zeta_0^1} = 0. \tag{4.3}$$

This implies that ζ_1^2 is semi-basic and

$$\zeta_1^2 = A\zeta_0^1 + \mathrm{i}b\overline{\zeta_0^1} + C\alpha_0^2, \tag{4.4}$$

where A and C are complex-valued functions and b is real-valued. Moreover, an easy inspection shows that $b(uX) = b(u)/rs$, for every $u \in \mathscr{F}_1(f)$ and every $X \in K_1$.

Definition 4.1. The bundle $\mathscr{F}_1(f)$ induces on M an optical structure which is denoted by $Q(f) \to M$. The structure equations of $\mathrm{SL}(3, \mathbb{C})$ imply that $Q(f)$ is a shear-free geometry of rays without shear. We say that (M, f) is *non-degenerate* if b is everywhere non-zero. If (M, f) is non-degenerate, then the induced shear-free geometry of rays is twisting. The equality $b = 0$ implies the integrability of the optical structure. From now on we assume (M, f) to be non-degenerate. An optical structure $Q(M) \to M$ is called *locally embeddable* if, for every point $x \in M$ there exists an open neighbourhood U and a non-degenerate characteristic embedding $f: U \to \mathbb{RP}^5$ such that $Q(M)|_U = Q(f)$.

Theorem 4.2. *Let $Q(M) \to M$ be a real-analytic twisting shear-free geometry of rays. Then $Q(M)$ is locally embeddable.*

Proof. The proof is based on the Cartan–Kähler theorem and on Cartan's test of involution for quasi-linear Pfaffian systems. Therefore, we will assume some basic knowledge of the theory of exterior-differential systems. We refer the reader to refs. [1] and [4] for the definitions and fundamental theorems that we will use during the proof.

Since our theorem is local we may assume from the beginning the existence of a real-analytic transversal orientation $\boldsymbol{\Theta}$. Under these assumptions the normal parallelism $(\omega^0, ..., \omega^3)$ determined by $\boldsymbol{\Theta}$ is real-analytic.

Consider the manifold $Y = \mathrm{SL}(3, \mathbb{C}) \times M$ and the Pfaffian differential system generated by the equations

$$\begin{aligned}\theta^1 &=: \alpha^2_0 - \omega^3 = 0\,,\\ \theta^2 &=: \alpha^1_0 - \omega^1 = 0\,,\\ \theta^3 &=: \beta^1_0 - \omega^2 = 0\,,\\ \theta^4 &=: \beta^2_0 = 0\,,\end{aligned} \tag{4.5}$$

together with the independence condition

$$\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \beta^0_0 \neq 0\,. \tag{4.6}$$

For later convenience we set $\omega^3 = \eta^1$, $\omega^1 = \eta^2$, $\omega^2 = \eta^3$ and $\beta^0_0 = \eta^4$. Let us now complete $\theta^1, ..., \theta^4, \eta^1, ..., \eta^4$ with a set of one-forms $\{\pi^1, ..., \pi^{12}\}$ in order to get a global coframing on Y. This can be done by setting

$$\begin{aligned}&\pi^1 = \alpha^0_0\,, \quad &&\pi^2 = \alpha^0_1\,, \quad &&\pi^3 = \beta^0_1\,, \quad &&\pi^4 = \alpha^0_2\,,\\ &\pi^5 = \beta^0_2\,, &&\pi^6 = \alpha^1_1\,, &&\pi^7 = \beta^1_1\,, &&\pi^8 = \alpha^1_2\,,\\ &\pi^9 = \beta^1_2\,, &&\pi^{10} = \alpha^2_1\,, &&\pi^{11} = \beta^2_1\,, &&\pi^{12} = \omega^0\,.\end{aligned} \tag{4.7}$$

Using the structure equations of $\mathrm{SL}(3, \mathbb{C})$ and the structure equations of the normal parallelization we obtain

$$\begin{aligned}\mathrm{d}\theta^2 \equiv{} & -(\pi^1 + \pi^6) \wedge \eta^2 + (\pi^7 - \pi^{12}) \wedge \eta^3 - \pi^8 \wedge \eta^1\\ & + (s - B_2)\eta^3 \wedge \eta^1 - B_1 \eta^2 \wedge \eta^1 - \eta^3 \wedge \eta^4\,,\\ \mathrm{d}\theta^1 \equiv{} & -\pi^{10} \wedge \eta^2 + \pi^{11} \wedge \eta^3 - \pi^6 \wedge \eta^1 - 2\eta^2 \wedge \eta^3\,,\\ \mathrm{d}\theta^3 \equiv{} & -(\pi^7 - \pi^{12}) \wedge \eta^2 - (\pi^1 + \pi^6) \wedge \eta^3 - \pi^9 \wedge \eta^1\\ & -(s + B_2)\eta^2 \wedge \eta^1 + B_1 \eta^3 \wedge \eta^1 + \eta^2 \wedge \eta^4\,,\\ \mathrm{d}\theta^4 \equiv{} & -\pi^{11} \wedge \eta^2 - \pi^{10} \wedge \eta^3 - \pi^7 \wedge \eta^1\,,\end{aligned}$$

where $\equiv$ means equality mod$\{\theta^1, ..., \theta^4\}$. Then (4.5) is a *quasi-linear* Pfaffian differential system and its *reduced tableau matrix* is given by

$$\begin{bmatrix} -\pi^6 & -\pi^{10} & \pi^{11} & 0 \\ -\pi^8 & -(\pi^1+\pi^6) & (\pi^7-\pi^{12}) & 0 \\ -\pi^9 & -(\pi^7-\pi^{12}) & -(\pi^1+\pi^6) & 0 \\ -\pi^7 & -\pi^{11} & -\pi^{10} & 0 \end{bmatrix}.$$

Therefore, the *reduced Cartan characters* s'_1, s'_2, s'_3 and s'_4 of (4.5) are

$$s'_1=4\,,\qquad s'_2=4\,,\qquad s'_3=0\,,\qquad s'_4=4\,.$$

Next we need to compute the dimension t of the *linear variety of integral elements* of (4.5) over a generic point x of Y. This is given by the dimension of the affine subspace of all matrices $(P^i_a)_{i=1,\ldots,12;a=1,\ldots,4}$ such that

$$-\mu^{10}\wedge\eta^2+\mu^{11}\wedge\eta^3-\mu^6\wedge\eta^1-2\eta^2\wedge\eta^3=0\,,$$

$$-(\mu^1+\mu^6)\wedge\eta^2+(\mu^7-\mu^{12})\wedge\eta^3-\mu^8\wedge\eta^1+(s-B_2)\eta^3\wedge\eta^1$$
$$-B_1\eta^2\wedge\eta^1-\eta^3\wedge\eta^4=0\,,$$

$$-(\mu^7-\mu^{12})\wedge\eta^2-(\mu^1+\mu^6)\wedge\eta^3-\mu^9\wedge\eta^1-(s+B_2)\eta^2\wedge\eta^1$$
$$+B_1\eta^3\wedge\eta^1+\eta^2\wedge\eta^4=0\,,$$

$$-\mu^{11}\wedge\eta^2-\mu^{10}\wedge\eta^3-\mu^7\wedge\eta^1=0\,,$$

where $\mu^i=:P^i_a\eta^a|_x$, $i=1, \ldots, 12$. Clearly these equations must be satisfied at the point x. It is now an easy matter to check that the equations above give 20 independent linear equations on the P^i_a. Thus, the linear variety of the integral elements at the point x is a 28-dimensional affine subspace of $M^{12,4}(\mathbb{R})=\mathbb{R}^{48}$, for every point x of Y. It is important to note that among the equations of the integral element we have

$$P^{12}_4=1\,. \tag{4.8}$$

From all this we get

$$28=t=s'_1+2s'_2+3s'_3+4s'_4\,.$$

The Cartan test of involution implies that this last equality holds if and only if our quasi-linear Pfaffian system is in *involution*. Since we are working in the real-analytic category, the last conclusion implies that for every point $(p_0, A_0)\in Y=M\times\mathrm{SL}(3,\mathbb{C})$ there is a four-dimensional integral submanifold $X\subset Y$ passing through (p_0, A_0). Moreover, from (4.8) we have that the restriction of $\{\omega^0, \ldots, \omega^3\}$ along X is a global coframing. Therefore, the projection H of X onto M is a local diffeomorphism. Define f' to be the mapping of X into $\mathbb{RP}^5$ obtained by projecting X in $\mathrm{SL}(3,\mathbb{C})$ and then composing with the bundle map onto $\mathbb{P}^5\mathbb{R}$. Finally, if we take a small neighbourhood U of p_0 we may invert H and we may define an embedding f of U into $\mathbb{RP}^5$ by setting $f=f'H^{-1}$. Equations (4.5) imply that f is a characteristic embedding and the optical structure $Q(f)$ coincides with $Q(M)|_U$. □

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